ON A SUFFICIENT CONDITION FOR THE STABILITY OF THE TRIVIAL SOLUTION OF A SYSTEM OF TWO LINEAR DIFFERENTIAL EQUATIONS

(OB ODNOM DOSTATOCHNOM USLOVII USTOICHIVSTI TRIVIAL'NOGO RESHENIIA SISTEMY IZ DVUKH LINEINYKH DIFFERENTSIAL'NYKH URAVNENII)

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N. P. KUPTSOV
(Saratov)

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We consider the equation

$$\ddot{x} + p(t)\dot{x} + q(t)x = 0 \tag{1}$$

for which there are known a large number of various types of sufficient conditions for the stability of its trivial solution [1]. In particular, a simple and convenient criterion of the Liapunov type was established by Leonov [2] (see also [3,4]): if

$$q(t) > 0, \qquad p(t) + \frac{\dot{q}(t)}{2q(t)} \geqslant 0$$

then the solutions of Equation (1) is stable* relative to x.

In the present note there is derived a new sufficient condition for stability, which generalizes the above-mentioned criterion of Leonov.

Suppose we are given the system of linear differential equations

$$\dot{x} = a_{11}(t)x + a_{12}(t)y, \qquad \dot{y} = a_{21}(t)x + a_{22}(t)y \tag{2}$$

with piecewise continuous coefficients. Let us consider the quadratic form

$$U = A(t)x^2 + 2B(t)xy + C(t)y^2 \tag{3}$$

* In order to have stability with respect to $\dot{x}$ it is necessary to have some additional requirements (for example, the boundedness of $\dot{q}(t)$ on $(0, \infty)$ (see [4, pp. 372-373]).

whose coefficients satisfy the system of linear equations

$$\begin{aligned} \dot{A} &= -2a_{11}A - 2a_{21}B \\ \dot{B} &= -a_{12}A - (a_{11} + a_{22})B - a_{21}C \\ \dot{C} &= -2a_{12}B - 2a_{22}C \end{aligned} \tag{4}$$

with the initial conditions

$$A(0) = A_0 > 0, \qquad B(0) = B_0, \qquad C(0) = C_0 > 0, \qquad A_0C_0 - B_0^2 > 0 \tag{5}$$

Let us set $\Delta t = A(t)C(t) - B^2(t)$. Differentiating $\Delta(t)$ and taking account of (4), we obtain

$$\dot{\Delta}(t) = -2(a_{11} + a_{22})\Delta(t)$$

Whence

$$\Delta(t) = \Delta(0)\exp\left[-2\int_0^t (a_{11} + a_{22})\,d\tau\right] \tag{6}$$

Therefore, $A(t) > 0$, and $C(t) > 0$ when $t > 0$. From this it follows that for $t > 0$ the equation

$$A(t)x^2 + 2B(t)xy + C(t)y^2 = \text{const}$$

determines some ellipse in the xy-plane.

If we substitute a solution of (2) for x and y in Formula (3), then U will be independent of t (this fact can easily be verified by differentiating U with respect to t). Let us assume that along the given solution $x(t)$, $y(t)$ of the system (2), the value of U is equal to U_0. This means that the point $x(t)$, $y(t)$ lies on the ellipse

$$A(t)x^2 + 2B(t)xy + C(t)y^2 = U_0$$

The point with maximum abscissa (ordinate) on this ellipse has the coordinates

$$\left\{\sqrt{\frac{U_0C(t)}{\Delta(t)}},\ -B(t)\sqrt{\frac{U_0}{C(t)\Delta(t)}}\right\} \qquad \left(\left\{-B(t)\sqrt{\frac{U_0}{C(t)\Delta(t)}},\ \sqrt{\frac{U_0A(t)}{\Delta(t)}}\right\}\right)$$

Therefore

$$|x(t)| \leqslant \sqrt{U_0\frac{C(t)}{\Delta(t)}}, \qquad |y(t)| \leqslant \sqrt{U_0\frac{A(t)}{\Delta(t)}} \tag{7}$$

For the boundedness of $x(t)$ on $(0, \infty)$ it is, therefore, sufficient that the expression $C(t)/\Delta(t)$ be bounded on $(0, \infty)$. The system (2) is linear. Hence, we draw the following conclusion on the basis of the preceding statements: in order that the trivial solution of the system (2) be stable relative to x, it is sufficient that $C(t)/\Delta(t)$ be bounded on $(0, \infty)$.

It is not difficult to prove that the boundedness of $C(t)/\Delta(t)$ on $(0, \infty)$ is also necessary for the stability of the trivial solution of (2) relative to x. In fact, suppose that we have this stability. Then there exists a constant M such that for every solution of the system (2) which satisfies the condition

$$A_0 x^2(0) + 2B_0 x(0)\, y(0) + C_0 y^2(0) = 1 \tag{8}$$

it is true that $|x(t)| \leqslant M$ for all $t > 0$.

A solution of the system (2) which satisfies the conditions

$$x(t_0) = \sqrt{\frac{C(t_0)}{\Delta(t_0)}}, \qquad y(t_0) = -B(t_0)\sqrt{\frac{1}{C(t_0)\,\Delta(t_0)}}$$

will also satisfy the equation $A(t_0)x^2(t_0) + 2B(t_0)x(t_0)y(t_0) + C(t_0)y^2(t_0) = 1$.

Since the function U is constant along every solution of Equation (2), we obtain (8)

$$A_0 x^2(0) + 2B_0 x(0)\, y(0) + C_0 y^2(0) = 1.$$

Therefore

$$x(t_0) = \sqrt{\frac{C(t_0)}{\Delta(t_0)}} \leqslant M$$

The arbitrariness of t_0 is still to be taken into account.

In a similar way we can deduce from (7) that a necessary and sufficient condition for the stability of the trivial solution of the system (2) relative to y is the boundedness of $A(t)/\Delta(t)$ on $(0, \infty)$. Next, suppose that $s(t)$ is an arbitrary function which is positive and has a continuous derivative on $(0, \infty)$. Let us set

$$\lambda(t) = \frac{1}{\sqrt{s(t)}} \exp \int_0^t \left\{ a_{11} + a_{22} - \left[\left(\frac{\dot{s}}{2s} + a_{11} - a_{22} \right)^2 + \frac{(a_{21} + s a_{12})^2}{s} \right]^{1/2} \right\} d\tau \tag{9}$$

Then

$$\frac{\dot{\lambda}}{\lambda} = -\frac{\dot{s}}{2s} + a_{11} + a_{22} - \left[\left(\frac{\dot{s}}{2s} + a_{11} - a_{22} \right)^2 + \frac{(a_{21} + s a_{12})^2}{s} \right]^{1/2} \tag{10}$$

For the purpose of simplifying the formulas we set

$$\alpha = -\dot{\lambda} + 2\lambda a_{11}, \qquad \beta = -\lambda(a_{21} + s a_{12}), \qquad \gamma = -\dot{\lambda} s - \lambda \dot{s} + 2\lambda s a_{22} \tag{11}$$

From (10) we easily obtain

$$\alpha = \lambda \left\{ \left[\left(\frac{\dot{s}}{2s} + a_{11} - a_{22} \right)^2 + \frac{1}{s} (a_{21} + s a_{12})^2 \right]^{1/2} + \frac{\dot{s}}{2s} + a_{11} - a_{22} \right\} \geqslant 0$$

$$\gamma = \lambda s \left\{ \left[\left(\frac{\dot{s}}{2s} + a_{11} - a_{22} \right)^2 + \frac{1}{s} (a_{21} + s a_{12})^2 \right]^{1/2} - \frac{\dot{s}}{2s} - a_{11} + a_{22} \right\} \geqslant 0 \tag{12}$$

$$\alpha\gamma - \beta^2 = 0$$

Let $J(t) = \lambda(A + sC)$. Taking into account (4), we have

$$\dot{J}(t) = \dot{\lambda}(A + sC) + \lambda(\dot{A} + s\dot{C} + \dot{s}C) = -\alpha A + 2\beta B - \gamma C$$

If the function $\alpha(t)$ vanishes for some value of t, then $\beta(t)$ vanishes also at this point (see (12)). In this case $J(t) = -\gamma C \leqslant 0$. It is easily verified that for $\alpha(t) \neq 0$, the following equation holds:

$$\dot{J}(t) = -\frac{(A\alpha - B\beta)^2}{A\alpha} - \frac{\gamma}{A}(AC - B^2) \leqslant 0$$

We have thus proved that $\dot{J}(t) \leqslant 0$ when $t > 0$. Hence, $J(t) \leqslant J(0)$. Taking into account (6) and (9), we obtain

$$\frac{C(t)}{\Delta(t)} \leqslant \frac{J(t)}{\lambda(t)\, s(t)} \frac{1}{\Delta(0)} \exp\left[2 \int_0^t (a_{11} + a_{22})\, d\tau\right] \leqslant$$

$$\leqslant \frac{J(0)}{\Delta(0)\sqrt{s(0)}} \exp \int_0^t \left\{ \left[\left(\frac{\dot{s}}{2s} + a_{11} - a_{22} \right)^2 + \frac{1}{s} (a_{21} + s a_{12})^2 \right]^{1/2} - \frac{\dot{s}}{2s} + a_{11} + a_{22} \right\} d\tau$$

This inequality implies the following theorem.

Theorem. If there exist a positive function $s(t)$ which has a continuous derivative on $(0, \infty)$, and a constant M such that

$$\int_0^t \left\{ \left[\left(\frac{\dot{s}}{2s} + a_{11} - a_{22} \right)^2 + \frac{1}{s} (a_{11} + s a_{12})^2 \right]^{1/2} - \frac{\dot{s}}{2s} + a_{11} + a_{22} \right\} d\tau \leqslant M \tag{13}$$

for all $t > 0$, then the trivial solution of the system (2) is stable relative to x.

In an analogous way one can establish a sufficient condition for the stability of the trivial solution of the system (2) relative to y:

$$\int_0^t \left\{ \left[\left(\frac{\dot{s}}{2s} + a_{11} - a_{22} \right)^2 + \frac{1}{s} (a_{21} + s a_{12})^2 \right]^{1/2} + \frac{\dot{s}}{2s} + a_{11} + a_{22} \right\} d\tau \leqslant M \tag{14}$$

Let us consider some particular criteria which can be deduced from the theorem just proved.

1. If a_{21}/a_{12} is negative and has a continuous derivative on $(0, \infty)$,

then one can set $s = -a_{21}/a_{12}$. The sufficient condition for stability relative to x of the trivial solution of the system (2) can be written in the form

$$\int_0^t \left[\left| a_{11} - a_{22} + \frac{a_{12}}{2a_{21}} \frac{d}{d\tau}\left(\frac{a_{21}}{a_{12}}\right) \right| + a_{11} + a_{22} - \frac{a_{12}}{2a_{21}} \frac{d}{d\tau}\left(\frac{a_{21}}{a_{12}}\right) \right] d\tau \leqslant M$$

In particular, if one reduces Equation (1) to the system (2), then the last condition takes the form

$$q(t) > 0, \qquad \int_0^\infty \left[\left| p + \frac{\dot q}{2q} \right| - \left(p + \frac{\dot q}{2q} \right) \right] d\tau < \infty$$

This sufficient condition of stability relative to x is a generalization of the conditions of Leonov mentioned at the beginning of this note.

2. For the differential equation (1) the inequality (13) can be written in the form

$$\int_0^t \left\{ \left[\left(p + \frac{\dot s}{2s} \right)^2 + \frac{(q-s)^2}{s} \right]^{1/2} - p - \frac{\dot s}{2s} \right\} d\tau \leqslant M$$

This gives rise to the following criterion: if there exists a positive function $s(t)$, which has a continuous derivative on $(0, \infty)$ and satisfies the condition

$$\int_0^\infty \frac{|q-s|}{\sqrt{s}}\, dt < \infty, \qquad \int_0^\infty \left[\left| p + \frac{\dot s}{2s} \right| - \left(p + \frac{\dot s}{2s} \right) \right] dt < \infty$$

then the trivial solution of the system (2) is stable relative to x.

3. If there exists a constant $\sigma > 0$ satisfying the requirement

$$\int_0^\infty |a_{21} + \sigma a_{12}|\, dt < \infty$$

then by setting $s(t) = \sigma$ we obtain a sufficient condition for the stability of the trivial solution of the system (2) relative to x and y in the form

$$\int_0^t (|a_{11} - a_{22}| + a_{11} + a_{22})\, d\tau \leqslant M \text{ when } t > 0 \qquad (M \text{ is an arbitrary constant})$$

The formulation of criteria of stability relative to y, which are analogous to those in 1 and 2, does not present any difficulties.

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